

Polynomial-Time Approximation Schemes for Circle and Other Packing Problems*

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Abstract

We give an asymptotic approximation scheme (APTAS) for the problem of packing a set of circles into a minimum number of unit square bins. To obtain rational solutions, we use augmented bins of height $1 + \gamma$, for some arbitrarily small number $\gamma > 0$. Our algorithm is polynomial on $\log 1/\gamma$, and thus γ is part of the problem input. For the special case that γ is constant, we give a (one dimensional) resource augmentation scheme, that is, we obtain a packing into bins of unit width and height $1 + \gamma$ using no more than the number of bins in an optimal packing. Additionally, we obtain an APTAS for the circle strip packing problem, whose goal is to pack a set of circles into a strip of unit width and minimum height. These are the first approximation and resource augmentation schemes for these problems.

Our algorithm is based on novel ideas of iteratively separating small and large items, and may be extended to a wide range of packing problems that satisfy certain conditions. These extensions comprise problems with different kinds of items, such as regular polygons, or with bins of different shapes, such as circles and spheres. As an example, we obtain APTAS's for the problems of packing d -dimensional spheres into hypercubes under the L_p -norm.

1 Introduction

In the *circle bin packing problem*, we are given a list \mathcal{C} of n circles identified by their indexes, $\mathcal{C} = \{1, \dots, n\}$, where circle i has radius $r_i \leq 1/2$, for $1 \leq i \leq n$, and an unlimited number of identical square bins of unit side. A packing is a non-overlapping placement of circles into a set of bins, such that every circle is fully contained in a bin. Here, one circle i is packed into one bin if we can associate coordinates x_i, y_i to its center, such that $r_i \leq x_i, y_i \leq 1 - r_i$, and for every other circle j with coordinates x_j, y_j that is packed into the same bin, we have $(x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i + r_j)^2$. The objective is to find a packing of \mathcal{C} into a minimum

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number of bins. In the *circle strip packing problem*, that is also known as *circle two-dimensional open dimension problem* [22, 35], the set of circles \mathcal{C} must be packed into a strip of unit width and unbounded height, and the objective is to obtain a packing of minimum height.

There are several results in the literature for packing problems involving circles, that are tackled using different methods, such as continuous and nonlinear systems, and discrete methods [8]. A large portion of the works deals with the problem of maximizing the number of circles of a given radius in a square [33]. In the context of origami design, Demaine *et al.* [14] proved that the decision problem that asks whether a set of circles can be packed into a unit square or into an equilateral triangle is NP-hard. Therefore, the circle bin packing problem and the circle strip packing problem are also NP-hard.

We are interested in the design of approximation algorithms for the circle bin packing and the circle strip packing problems. As it is usual for packing problems, the measure we look for is the asymptotic performance. Given an algorithm \mathcal{A} , and a problem instance I , we denote by $\mathcal{A}(I)$ the value of the solution produced by \mathcal{A} , and by $\text{OPT}(I)$ the value of an optimum solution for I . For some $\alpha \geq 1$, we say that a polynomial-time algorithm \mathcal{A} (for a minimization problem) is an *asymptotic α -approximation algorithm* if, for every instance I , we obtain $\mathcal{A}(I) \leq \alpha \text{OPT}(I) + O(1)$. Also, a family of polynomial-time algorithms $\{\mathcal{A}_\varepsilon\}$ is said to be an *asymptotic polynomial-time approximation scheme* (APTAS) if, for every instance I , and fixed $\varepsilon > 0$, we have $\mathcal{A}_\varepsilon(I) \leq (1 + \varepsilon) \text{OPT}(I) + O(1)$. If the constant term $O(1)$ is omitted from the definitions, that we say that \mathcal{A} is an *α -approximation algorithm*, and $\{\mathcal{A}_\varepsilon\}$ is a *polynomial-time approximation scheme* (PTAS), respectively.

1.1 Our results and techniques

In this paper, we present a new algorithm for a series of packing problems. We give APTAS's for both the circle bin packing, and the circle strip packing problems. The bin packing problem is considered when we allow the use of augmented bins of unit width and height $1 + \gamma$, for some arbitrarily small $\gamma > 0$. This relaxation is necessary due to numeric concerns, as the coordinates for the circles' centers obtained by our algorithms are given by roots of polynomial equations, that are possibly irrational. In order to obtain rational solutions, we approximate the coordinates and increase the height of a bin slightly. To the best of our knowledge, these are the first approximation guarantees for these natural problems.

We highlight that the time complexity of the algorithm depends polynomially on $\log 1/\gamma$, so the value of parameter γ may be given as part of the problem instance. For the case that a bin is enlarged by an arbitrary but constant value $\varepsilon > 0$, we give a resource augmentation scheme in one dimension, that is, for any constant $\varepsilon > 0$, we develop a polynomial-time algorithm \mathcal{A}_ε that returns a packing of \mathcal{C} into bins of unit width and height $1 + \varepsilon$ with size $\mathcal{A}_\varepsilon(\mathcal{C}) \leq \text{OPT}(\mathcal{C})$, where $\text{OPT}(\mathcal{C})$ is the optimal value of the problem without resource augmentation.

Although the algorithm presented here is described only for the circle bin packing, it can be seen as a *unified framework*, and extends to a wide range of different packing problems, such as the bin packing of ellipses, regular polygons, and many others. Indeed, as an illustrative example, we show how to generalize our results to the bin packing and to the strip packing of d -dimensional L_p -norm spheres.

Our algorithm uses some techniques that have appeared in the literature in several and new interesting ways. As usual in the packing of rectangular items, our algorithm distinguishes between “large” and “small” items. However, this distinction is dynamic, so that an item can be considered small in one iteration, but large in a later one. There are two main novel ideas that differ from the approaches for rectangle packing, and that are needed for the circle packing. First, we reduce the packing of large circles to the problem of solving a semi-algebraic system,

that can be done with the aid of standard quantifier elimination algorithms from algebra. This helped us to avoid the use of combinatorial brute-force algorithms that are based on discretization, which would depend exponentially on $\log 1/\gamma$. Second, to pack small items, we cut the free space of previous packings into smaller sub-bins, and use the algorithm for large items, recursively.

1.2 Related works

In the literature of approximation algorithms, the majority of the works consider the packing of simple items into larger recipients, such as rectangular bins and strips. Most of the works which give approximation guarantees are interested in rectangular items or d -dimensional boxes. The packing problems involving circles are mainly considered through heuristics, or numerical methods, and, to our knowledge, there is no approximation algorithm for the circle bin packing or for the circle strip packing problems. On the practical side, packing problems have numerous applications, such as packaging of boxes in containers, or cutting of material. An application of circular packing is, for example, obtaining a maximal coverage of radio towers in a geographical region [33].

The problem of finding the densest packing of equal circles into a square has been largely investigated using many different optimization methods. For an extensive book on this problem, and corresponding approaches, see [33]. The case of circles of different sizes is considered in [17], where heuristics, such as genetic algorithms, are proposed to pack circles into a rectangular container. The circle strip packing has been considered using many approaches, such as branch-and-bound, metaheuristics, etc. For a broad list of algorithms for the circle strip packing, and related circle packing problems, see [22] and references therein.

For the problem of packing rectangles into rectangular bins, Chung *et al.* [10] presented a hybrid algorithm, called HFF, combining a one-dimensional packing algorithm (FFD) with a strip packing algorithm (FFDH) to obtain an algorithm with asymptotic approximation ratio 2.125. Caprara [9] proved that the asymptotic approximation ratio of the algorithm HFF is 2.077, and also presented a better algorithm, with approximation ratio 1.691. Bansal *et al.* [2] improved this ratio with a probabilistic algorithm, that can be derandomized, with asymptotic approximation ratio that can be as close to 1.525 as desired. Recently, Bansal and Khan [5] gave an asymptotic 1.405-approximation. Considering non-asymptotic approximation ratio, Harren and van Stee [21] showed that HFF has ratio 3, and presented an algorithm with ratio 2 [20]. For the bin packing of d -dimensional cubes, Kohayakawa *et al.* [25] showed an asymptotic ratio of $2 - (2/3)^d$, later improved to an APTAS by Bansal *et al.* [3]. For a survey on bin packing, see [12].

The first approximation algorithm for the rectangle strip packing problem was proposed by Baker *et al.* [1]. They presented the so called BL (Bottom-Leftmost) algorithm, and showed that it has approximation ratio 3. For the special case when all items are squares, the approximation ratio of BL algorithm is at most 2. Coffman *et al.* [11] presented three algorithms, denoted by NFDH (Next Fit Decreasing Height), FFDH (First Fit Decreasing Height), and SF (Split Fit), with asymptotic approximation ratios of 2, 1.7, and 1.5, respectively. They also showed that, when the items are squares, FFDH has an asymptotic approximation ratio of 1.5, and, when all items have width at most $1/m$, the algorithms FFDH and SFFDH have asymptotic approximation ratios of $(m+1)/m$ and $(m+2)/(m+1)$, respectively. The best known ratio for the problem was obtained by Kenyon and Rémila [24], who presented an asymptotic approximation scheme. Considering non-asymptotic approximation algorithms, Sleator [31] presented a ratio 2.5. This result was improved to 2 independently by Schiermeyer [30] and Steinberg [32], then to 1.9396 by Harren and van Stee [20], and finally to $5/3 + \varepsilon$ by Harren *et al.* [19].

For the 3-dimensional strip packing problem, whose items are boxes, Li and Cheng [27] were the first to present an asymptotic 3.25-approximation algorithm. Their algorithm was shown to have approximation ratio 2.67 [29], $2 + \varepsilon$ [23] and finally 1.69 [4]. Bansal *et al.* [3] showed that there is no asymptotic approximation scheme for the rectangle bin packing problem, which implies that there is no APTAS for the 3-dimensional strip packing problem. When the items are cubes, the first specialized algorithm was shown to have asymptotic ratio of 2.6875 [27], and the best result is an asymptotic approximation scheme due to Bansal *et al.* [4].

Organization The remainder of this paper is organized as follows. In Section 2, we discuss how to decide whether a set of n circles can all be packed in a rectangular bin using algebraic quantifier elimination. In Section 3, we give approximation algorithms for the case of “large” circles. In Section 4, we present APTAS’s for the circle bin packing problem, and for the circle strip packing problem. In Section 5, we give a resource augmentation scheme for the circle bin packing. In Section 6, we generalize the results of Section 4 to the case of multidimensional spheres, and to the case of items and bins with different shapes. In Section 7, we give the final remarks.

2 Circle Packing Through Algebraic Quantifier Elimination

Throughout this paper, we will consider instances for circle packing problems as in the following definition.

Definition 1. A triple (\mathcal{C}, w, h) is an instance for the circle packing problem if $h, w \in \mathbb{Q}_+$, and $\mathcal{C} = \{1, \dots, n\}$ is a set of circles, such that each circle i has radius $r_i \in \mathbb{Q}_+$, with $2r_i \leq \min\{w, h\}$, for $1 \leq i \leq n$.

In this section, we consider the following *circle packing decision problem*. An instance for this problem is a triple (\mathcal{C}, w, h) , and the objective is to decide whether the circles in \mathcal{C} can be packed into a bin of size $w \times h$, that is, a rectangle of width w and height h . In the case of a positive answer, a realization of the packing should also be returned. More precisely, for each circle i , with $1 \leq i \leq n$, we want to find a point $(x_i, y_i) \in \mathbb{R}_+^2$ that represents the center of i in a rectangle whose bottom-left and top-right corners correspond to points $(0, 0)$ and (w, h) , respectively.

The circle packing decision problem can be equivalently formulated as deciding whether there are real numbers $x_i, y_i \in \mathbb{R}_+$, for $1 \leq i \leq n$, that satisfy the constraints

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i + r_j)^2 \quad \text{for } 1 \leq i < j \leq n, \quad (1)$$

$$r_i \leq x_i \leq w - r_i \quad \text{for } 1 \leq i \leq n, \text{ and} \quad (2)$$

$$r_i \leq y_i \leq h - r_i \quad \text{for } 1 \leq i \leq n. \quad (3)$$

The set of constraints (1) guarantees that no two circles intersect, and the sets of constraints (2) and (3) ensure that each circle has to be packed entirely in the rectangle that expands from the origin $(0, 0)$ to the point (w, h) .

We observe that the set of solutions that satisfy (1)-(3) is a semi-algebraic set in the field of the real numbers. Thus, the circle packing decision problem corresponds to deciding whether this semi-algebraic set is not empty. We also can rewrite the constraints in (1)-(3) as $f_i(x_1, y_1, \dots, x_n, y_n) \geq 0$, for $1 \leq i \leq s$, where s is the total number of constraints, and $f_i \in \mathbb{Q}[x_1, y_1, \dots, x_n, y_n]$ is a polynomial with rational coefficients. Then, the circle packing problem is equivalent to deciding the truth of the formula

$$(\exists x_1)(\exists y_1) \dots (\exists x_n)(\exists y_n) \bigwedge_{i=1}^s f_i(x_1, y_1, \dots, x_n, y_n) \geq 0. \quad (4)$$

We can use any algorithm for the more general quantifier elimination problem to decide this formula. There are several algorithms for this problem, such as the algorithm of Tarski-Seidenberg Theorem [34], that is not elementary recursive, or the Cylindrical Decomposition Algorithm [13], that is doubly exponential in the number of variables. Since the formula corresponding to the circle packing problem contains only one block of variables (of existential quantifiers), we can use faster algorithms for the corresponding algebraic existential problem, such as the algorithms of Grigor'ev and Vorobjov [18], or of Basu *et al.* [6]. For an extensive list of algorithms for real algebraic geometry, see [7].

Sampling points of the solution Any of the algorithms above receiving formula (4) as input will return “true” if, and only if, the circles in \mathcal{C} can be packed into a bin of size $w \times h$. When the answer is “true”, we are also interested in a realization of such packing. The algorithms in [18, 6] are based on critical points, that is, they also return a finite set of points that meets every semi-algebraic connected component of the semi-algebraic set. Thus, a realization of the packing can be obtained by choosing one of such points (that is a point that corresponds to a connected component where all polynomials f_i , $1 \leq i \leq s$, are nonnegative).

Typically, a sample point is represented by a tuple $(f(x), g_0(x), \dots, g_k(x))$ of $k+2$ univariate polynomials with coefficients in \mathbb{Q} , where k is the number of variables, and the value of the i -th variable is $g_i(x)/g_0(x)$ evaluated at a real root of $f(x)$. Since a point in a semi-algebraic set may potentially be irrational, we use the algorithm of Grigor'ev and Vorobjov [18], for which we have $g_0(x) = 1$, and thus an approximate rational solution of arbitrary precision can be readily obtained. In particular, the theorem given in [18] implies the following result.

Corollary 1. *Let $f_1, \dots, f_s \in \mathbb{Q}[x_1, y_1, \dots, x_n, y_n]$ be polynomials with coefficients of bit-size at most m , and maximum degree 2. There is an algorithm that decides the truth of formula (4), with running time $m^{O(1)}s^{O(n^2)}$. In the case of affirmative answer, then the algorithm also returns polynomials $f, g_1, h_1, \dots, g_n, h_n \in \mathbb{Q}[x]$ with coefficients of bit-size at most $m^{O(1)}s^{O(n)}$, and maximum degree $s^{O(n)}$, such that for a root x of $f(x)$, the assignment $x_1 = g_1(x), y_1 = h_1(x), \dots, x_n = g_n(x), y_n = h_n(x)$ is a realization of (4). Moreover, for any rational $\alpha > 0$, we can obtain $x'_1, y'_1, \dots, x'_n, y'_n \in \mathbb{Q}$, such that $|x'_i - x_i| \leq \alpha$ and $|y'_i - y_i| \leq \alpha$, $1 \leq i \leq n$, with running time $(\log(1/\alpha)m)^{O(1)}s^{O(n^2)}$.*

3 Approximate Bin Packing of Large Circles

In this section, we consider the special case of the circle bin packing problem in which the minimum radius of a circle is at least a constant. For this case, the maximum number of circles that fit in a bin is constant, so we can use the algorithm of Corollary 1 to decide in constant time whether a given set of circles can be packed into a bin. Since Corollary 1 only gives us rational solutions that are close to real-valued packings, we will first transform an approximate packing into a non-overlapping packing in an augmented bin.

Later, we obtain a PTAS for the special case of the circle bin packing problem with large items.

3.1 Transforming approximate packings

We start with the next definition to deal with approximate circle bin packings. In the following, we denote by $\text{dist}(p, q)$ the Euclidean distance between two points p, q of the 2-dimensional space.

Definition 2. Given an instance (\mathcal{C}, w, h) for the circle packing problem, and a number $\varepsilon \geq 0$, we say that a set of points $p_i = (x_i, y_i)$, for $1 \leq i \leq n$, is an ε -packing of \mathcal{C} into a rectangular bin of size $w \times h$, if the following hold:

$$\text{dist}(p_i, p_j) \geq r_i + r_j - \varepsilon \geq 0 \quad \text{for } 1 \leq i < j \leq n, \quad (5)$$

$$r_i - \varepsilon \leq x_i \leq w - r_i + \varepsilon \quad \text{for } 1 \leq i \leq n, \text{ and} \quad (6)$$

$$r_i - \varepsilon \leq y_i \leq h - r_i + \varepsilon \quad \text{for } 1 \leq i \leq n. \quad (7)$$

We adopt the following strategy to fix intersections of an approximate bin packing: (a) first, we shift the x -coordinate of all circles that intersect the left or right border until they are fully contained in the bin, (b) then, we iteratively lift each circle in order of the y -coordinate by an appropriate distance so that it does not intersect any of the circles considered in previous iterations. First, the next lemma bounds the distance that one circle needs to be raised to avoid intersection with lower circles in an εh -packing. Then, Lemma 2 transforms an approximate packing into a packing with bins of augmented height.

Lemma 1. Let r_1, r_2, h, ε be positive numbers such that $\varepsilon h \leq r_1 + r_2 \leq h$, and $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ be points in \mathbb{R}^2 . If $y_1 \geq y_2$, $\text{dist}(p_1, p_2) \geq r_1 + r_2 - \varepsilon h$, and $p'_1 = (x_1, y_1 + \sqrt{2\varepsilon h})$, then $\text{dist}(p'_1, p_2) \geq r_1 + r_2$.

Proof. By direct calculation,

$$\begin{aligned} \text{dist}(p'_1, p_2) &= \sqrt{(x_1 - x_2)^2 + (y_1 + \sqrt{2\varepsilon h} - y_2)^2} \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2\sqrt{2\varepsilon h}(y_1 - y_2) + 2\varepsilon h^2} \\ &= \sqrt{\text{dist}(p_1, p_2)^2 + 2\sqrt{2\varepsilon h}(y_1 - y_2) + 2\varepsilon h^2} \\ &\geq \sqrt{(r_1 + r_2 - \varepsilon h)^2 + 2\varepsilon h^2} \\ &= \sqrt{(r_1 + r_2)^2 - 2\varepsilon h(r_1 + r_2) + \varepsilon^2 h^2 + 2\varepsilon h^2} \\ &\geq r_1 + r_2, \end{aligned}$$

where the last inequality follows from $r_1 + r_2 \leq h$. \square

Lemma 2. Given an instance (\mathcal{C}, w, h) for the circle packing problem, and a corresponding εh -packing of \mathcal{C} into a bin of size $w \times h$ for some $\varepsilon > 0$, we can find a packing of \mathcal{C} into a bin of size $w \times (1 + n\sqrt{6\varepsilon})h$ in linear time.

Proof. For $1 \leq i \leq n$, let $p_i = (x_i, y_i)$ be the center of circle i corresponding to the εh -packing. We start by modifying the given εh -packing to obtain a $3\varepsilon h$ -packing into a bin of size $w \times (h + 2\varepsilon h)$, with the additional property that no circle intersects a border of such rectangular bin. For each $1 \leq i \leq n$, let $p'_i = (x'_i, y'_i)$ be the center of circle i in the modified packing. The y -coordinate is defined as $y'_i = y_i + \varepsilon$, and the x -coordinate is defined as: $x'_i = x_i$ if i does not intersect the left or right border; $x'_i = r_i$ if i intersects the left border; and $x'_i = w - r_i$ if i intersects the right border (notice that i cannot intersect both the left and the right borders, since $2r_i \leq w$). Clearly, the definition of the centers guarantees that no circle intersects any border of the augmented bin. To see that the set of points p'_i is a $3\varepsilon h$ -packing, just note that any two circles i, j are lifted by the same distance, so by the triangle inequality $\text{dist}(p'_i, p'_j) \geq \text{dist}(p_i, p_j) - 2\varepsilon h \geq r_i + r_j - 3\varepsilon h$.

Now, we transform the $3\varepsilon h$ -packing into a packing in a bin of width w and height $h + 2\varepsilon h + (n - 1)\sqrt{6\varepsilon}h \leq (1 + n\sqrt{6\varepsilon})h$. We can assume, without loss of generality, that circles $1, \dots, n$ are ordered in nondecreasing order of the y -coordinate of their centers p'_i . For

every circle i , $1 \leq i \leq n$, define its new center as $p_i'' = (x_i', y_i' + (i-1)\sqrt{6\varepsilon}h)$. Notice that the y -coordinate of the last circle is increased by $(n-1)\sqrt{6\varepsilon}h$, and thus no circle intersects any of the borders. To complete the proof, we show that no two circles i and j , with $1 \leq j < i \leq n$, intersect:

$$\begin{aligned} \text{dist}(p_i'', p_j'') &= \text{dist}((x_i', y_i' + (i-1)\sqrt{6\varepsilon}h), (x_j', y_j' + (j-1)\sqrt{6\varepsilon}h)) \\ &= \text{dist}((x_i', y_i' + (i-j)\sqrt{6\varepsilon}h), (x_j', y_j')) \\ &\geq \text{dist}((x_i', y_i' + \sqrt{6\varepsilon}h), (x_j', y_j')) \geq r_i + r_j, \end{aligned}$$

where the last inequality follows from Lemma 1, and the fact that we had a $3\varepsilon h$ -packing. \square

3.2 A PTAS for large items

The optimal value for the circle bin packing problem is defined next.

Definition 3. *Given an instance (\mathcal{C}, w, h) for the circle bin packing, we denote by $\text{OPT}_{w \times h}(\mathcal{C})$ the minimum number of rectangular bins of size $w \times h$ that are necessary to pack \mathcal{C} .*

We will obtain an approximation algorithm for the bin packing of large circles, that is, assuming that the radius of each circle is greater than a given constant. In this case, we can use an algorithm similar to that by Fernandez De La Vega and Lueker [16] to obtain a bin packing of large circles. In this case, the maximum number of circles that fit in a bin is at most a constant, M , so we can partition the set of circles into a small number of groups with approximate sizes, and enumerate all patterns of groups with no more than M circles. Then, we may apply the algorithm of Corollary 1 to list which patterns correspond to feasible packings, and use integer programming in fixed dimension to find out how many bins of each pattern are necessary to cover all circles.

We split the algorithm in two parts. First, Lemma 3 considers the special case where the number of different radii is bounded by a constant, and obtain a bin packing using at most the optimal number of bins, $\text{OPT}_{w \times h}(\mathcal{C})$. Then, Theorem 1 reduces the general case to the case of bounded number of radii, and obtain a bin packing using at most an additional ε fraction of $\text{OPT}_{w \times h}(\mathcal{C})$. In the following, we will denote the area of the circle of radius r by $\text{Area}(r)$.

Lemma 3. *Let (\mathcal{C}, w, h) be an instance of the circle bin packing, such that $w, h \in O(1)$, $\min_{1 \leq i \leq n} r_i \geq \delta$, and $|\{r_1, \dots, r_n\}| \leq K$, for constants K and δ . For any given number $\gamma > 0$, we can obtain a packing of \mathcal{C} into at most $\text{OPT}_{w \times h}(\mathcal{C})$ rectangular bins of size $w \times (1 + \gamma)h$ in polynomial time.*

Proof. Notice that a bin of size $w \times h$ may contain at most $M = \lceil wh/\text{Area}(\delta) \rceil$ circles of \mathcal{C} . Consider an ordering of distinct radii $\bar{r}_1, \dots, \bar{r}_K$. We say that a vector of nonnegative integers $c = (c_1, \dots, c_K)$ with $\sum_{i=1}^K c_i \leq M$ is a configuration, and, for each $1 \leq i \leq K$, that c_i is the number of circles with radius \bar{r}_i of c . A configuration c is said to be feasible if there is a packing into a bin of size $w \times h$ containing all circles of c .

Let $\varepsilon = \gamma^2/(6M^2)$, and let $\alpha = \varepsilon h/4$. We enumerate each of the (at most M^K) configurations c , and use the algorithm of Corollary 1 to decide whether c is feasible. For each feasible c , we also obtain an approximate packing, such that each circle of c has rational center p' at distance at most 2α of the center p in the packing realization. Then, for any two circles, with centers p_1 and p_2 in the packing realization, and approximate rational centers p'_1 and p'_2 , we have $\text{dist}(p_1, p_2) - \text{dist}(p'_1, p'_2) \leq 4\alpha = \varepsilon h$. Thus, the obtained approximate packing is an εh -packing of circles in c . For each feasible packing, we use Lemma 2, and obtain a packing of the circles of c into a bin of width w and height $(1 + M\sqrt{6\varepsilon})h = (1 + \gamma)h$. Let \mathcal{X} be the set

of feasible configurations, and let n_i , for $1 \leq i \leq K$, denote the number of circles in \mathcal{C} with radius \bar{r}_i . Solving the following integer program, we obtain a bin packing of size $\text{OPT}_{w \times h}(\mathcal{C})$ that contains x_c bins of configuration c for each $c \in \mathcal{X}$.

$$\begin{aligned} & \text{minimize} && \sum_{c \in \mathcal{X}} x_c \\ & \text{subject to} && \sum_{c \in \mathcal{X}} c_i x_c \geq n_i, \quad 1 \leq i \leq K \\ & && x_c \in \mathbb{Z}_+ \quad c \in \mathcal{X} \end{aligned}$$

Since this program has a constant number of variables, and bit-size $O(\log n)$, it can be solved in $O(\log n)$ using fixed dimension integer programming [26, 15]. \square

Now, the following theorem gives a $(1 + \varepsilon)$ -approximation for the particular case in which circles' radii are greater than a constant.

Theorem 1. *Let (\mathcal{C}, w, h) be an instance of the circle bin packing, such that $w, h \in O(1)$, and $\min_{1 \leq i \leq n} r_i \geq \delta$, for some constant δ . For any given constant $\varepsilon > 0$, and number $\gamma > 0$, there is a polynomial-time algorithm that packs \mathcal{C} into at most $(1 + \varepsilon) \text{OPT}_{w \times h}(\mathcal{C})$ rectangular bins of size $w \times (1 + \gamma)h$.*

Proof. First, let $K = \lceil 2/(\varepsilon \text{Area}(\delta)) \rceil$. If $n \leq K$, then we use the algorithm of Lemma 3 on instance \mathcal{C} , and obtain a packing of \mathcal{C} into at most $\text{OPT}_{w \times h}(\mathcal{C})$ bins of size $w \times (1 + \gamma)h$. If $n > K$, then we let $Q = \lfloor \varepsilon n \text{Area}(\delta) \rfloor > 1$, and execute the following steps:

1. sort the circles in non-increasing order of radius;
2. partition \mathcal{C} in groups of up to Q consecutive circles greedily;
3. create an instance \mathcal{C}' by changing the radius of each circle in \mathcal{C} to the smallest radius of its group;
4. use the algorithm of Lemma 3 to find a packing of \mathcal{C}' .

We have obtained a packing P' of \mathcal{C}' of size $\text{OPT}_{w \times h}(\mathcal{C}')$ into rectangular bins of size $w \times (1 + \gamma)h$. Notice that, with exception of circles in the first group, every circle in \mathcal{C} can be mapped to a circle in \mathcal{C}' of non-smaller radius, thus we can obtain a packing for \mathcal{C} with the following steps: pack each circle in the first group in a new bin; for every other circle, pack at the position of the mapped circle in P' . Thus, we have obtained a packing of \mathcal{C} that uses at most $\text{OPT}_{w \times h}(\mathcal{C}') + Q \leq \text{OPT}_{w \times h}(\mathcal{C}) + \varepsilon n \text{Area}(\delta) \leq (1 + \varepsilon) \text{OPT}_{w \times h}(\mathcal{C})$ bins.

If $n \leq K$, then the number of different radii in \mathcal{C} is at most K , otherwise, if $n > K$, the number of different radii in \mathcal{C}' is at most $\lceil \frac{n}{Q} \rceil = \lceil \frac{n}{\lfloor \varepsilon n \text{Area}(\delta) \rfloor} \rceil \leq \lceil \frac{2n}{\varepsilon n \text{Area}(\delta)} \rceil = K$. In either case, using Lemma 3, we can conclude that the algorithm is polynomial. \square

4 An Asymptotic PTAS for Circle Bin Packing

In this section, we consider the bin packing problem of circles of any size. The main idea works as follows. First, we find a subset of circles with “intermediate” sizes, such that, after the removal of this set, we can divide the remaining circles into a sequence of groups where the smallest circle in a group i is much larger than the largest circle in the next group $i + 1$, and so on. The set of removed circles is chosen so that its area is small (*i.e.*, it is bounded by an ε fraction of the overall circles' area), and thus we can pack them into separate bins using any constant-ratio approximation algorithm. To pack the sequence of groups, we do the following: we pack the first group (of “large” circles) using the algorithm from Section 3, and obtain a

packing into bins of the original size; then, we consider sub-bins with a small fraction of the original size, and solve the problem of packing the remaining groups (of “small” circles) in such sub-bins recursively. To obtain a solution of the original problem, we place each obtained small bin into the free space of the packing obtained for large circles.

The key idea to obtain an APTAS is that, if the size of the small bins is much smaller than the size of large circles, then the waste of space in the packing of the large circles is proportional to a fraction of large circles’ area. Moreover, if the size of a such small bin is also much larger than the size of small circles, then restricting the packing of small circles to small bins does not increase much the cost of a solution.

4.1 The algorithm

In the following, if B is a circle or rectangle, then we denote by $\text{Area}(B)$ the area of B . Also, if D is a set, then $\text{Area}(D) = \sum_{B \in D} \text{Area}(B)$. We give first a formal description in Algorithm 1; an informal description is given thereafter.

Algorithm 1 *Circle bin packing algorithm*

Consider the parameters r and γ , such that r is a positive integer multiple of 3, and $\gamma > 0$. The algorithm receives an instance (\mathcal{C}, w, h) for the circle bin packing, and returns a packing of \mathcal{C} into a set of bins of size $w \times (1 + \gamma)h$. It is assumed that $w \leq h$, and hr/w is an integer.

1. Let $\varepsilon = 1/r$;
2. For every integer $i \geq 0$, define $G_i = \{j \in \mathcal{C} : \varepsilon^{2i}w \geq 2r_j > \varepsilon^{2(i+1)}w\}$;
3. For each $0 \leq j < r$, define $H_j = \{\ell \in G_i : i \equiv j \pmod{r}\}$;
4. Find an integer t such that $\text{Area}(H_t) \leq \varepsilon \text{Area}(\mathcal{C})$;
5. Place each circle of H_t into its bounding box, and pack the boxes in separate bins of size $w \times (1 + \gamma)h$ using NFDH strategy [28];
6. For every integer $j \geq 0$, define $S_j = \bigcup_{i=t+(j-1)r+1}^{t+jr-1} G_i$; (see Figure 1)
7. Define $w_0 = w$, $h_0 = h$, and $w_j = h_j = \varepsilon^{2(t+(j-1)r)+1}w$ for every $j \geq 1$;
8. Let $F_0 = \emptyset$;
9. For every $j \geq 0$:
 - (a) Use the algorithm of Theorem 1 to obtain a packing of circles S_j into bins of size $w_j \times (1 + \gamma)h_j$. Let P_j be the set of such bins;
 - (b) Let A_j be a set of $\max\{|P_j| - |F_j|, 0\}$ new empty bins of size $w_j \times (1 + \gamma)h_j$;
 - (c) Place each bin of P_j into one distinct bin of $F_j \cup A_j$;
 - (d) Set $F_{j+1} = \emptyset$, and $U_j = \emptyset$; (U_j is used only in the analysis)
 - (e) For each bin B of $F_j \cup A_j$:
 - Let V be the set of bins corresponding to the cells of the grid with cells of size $w_{j+1} \times (1 + \gamma)h_{j+1}$ over B ;
 - Add to F_{j+1} all bins in V that do not intersect any circle of S_j .
 - Add to U_j all bins in V that intersect a circle of S_j .
 - (f) If all circles are packed, go to step 10.
10. Place the bins A_0, A_1, \dots into the minimum number of bins of size $w \times (1 + \gamma)h$.

		$H_t:$
$S_0:$	G_0, \dots, G_{t-1}	G_t
$S_1:$	$G_{t+1}, G_{t+2}, \dots, G_{t+r-1}$	G_{t+r}
$S_2:$	$G_{t+r+1}, G_{t+r+2}, \dots, G_{t+2r-1}$	G_{t+2r}
$S_j:$	$G_{t+(j-1)r+1}, G_{t+jr+2}, \dots, G_{t+jr-1}$	G_{t+jr}
	\vdots	\vdots

Figure 1: Partitioning of the set of circles.

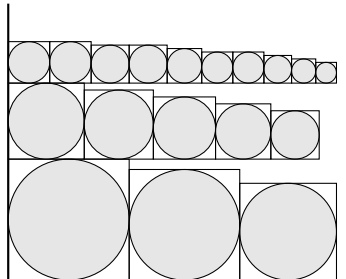
In step 2, we group the set of circles into disjoint subsets, G_i , with exponentially decreasing radii. Then, in step 3, these groups are joined into disjoint bunches, H_i , according to the remainder of the division of the group index by r . This induces the creation of r bunches, so in step 4 we may find a bunch H_t whose area corresponds to a fraction of at most $1/r = \varepsilon$ of the overall circles' area. In step 5, we pack the circles of H_t using the NFDH strategy, that is, we pack the circles H_t greedily in “shelves” and in decreasing order of radii, as in Figure 2.

In step 6, the remaining unpacked circles are joined in maximal sequences, S_j , of consecutive groups. This leads to a partition of \mathcal{C} into sets H_t, S_0, S_1, \dots . The considered groups and sets can be distributed over a table, as depicted in Figure 1, where each row corresponds to a sequence S_j , and the last column corresponds to bunch H_t .

For each j , we consider the subproblem of packing the circles of S_j into bins of size $w_j \times h_j$, as defined in step 7. This definition guarantees that the circles in S_j are large when compared to bins of size $w_j \times h_j$, and so we can use the algorithm for large circles of Theorem 1. Also, the bins considered in the next iteration have size $w_{j+1} \times h_{j+1}$, and are much smaller than the circles of S_j . Finally, since the circles in H_t are considered separately, each remaining unpacked circle (of S_{j+1}, S_{j+2}, \dots) fits in a bin of size $w_{j+1} \times h_{j+1}$.

In step 9, we solve each subproblem iteratively. In each iteration $j \geq 0$, the algorithm keeps a set F_j of free bins of size $w_j \times (1 + \gamma)h_j$ obtained from previous iterations. We obtain a packing of circles of S_j into a set of bins P_j . Then, we place such bins into the free space of F_j , or into additional bins A_j , if necessary. The set of sub-bins of $F_j \cup A_j$ of size $w_{j+1} \times (1 + \gamma)h_{j+1}$ that intersect circles of S_j are included in the set of used sub-bins U_j , and the remaining free sub-bins are saved in F_{j+1} for the next iteration. The algorithm finishes when all circles are considered, and the created bins A_0, A_1, \dots are combined into bins of size $w \times (1 + \gamma)h$ in step 10.

We remark that the assumption that hr/w is integer is without loss of generality. If this is not the case, then, after the first iteration of the algorithm, we could extend the height h_0 of bins



We pack each circle in the corresponding bounding box, and then use the NFDH strategy for rectangle bin packing.

Figure 2: Packing medium circles of H_t .

in P_0 to the next integer multiple of w/r . The final solution would have the property that circles packed in the extended area are completely packed in the top cells of size $w_1 \times h_1$. Thus, one can easily modify such a solution by moving these cells to at most $\lceil 1/r |P_0| \rceil \leq O(\varepsilon) \text{OPT}(\mathcal{C}) + 1$ new bins of size $w \times h$. For the sake of clarity, from now on, we assume that hr/w is integer.

4.2 Analysis

Now, we analyze the algorithm. The idea of the analysis is that the generated solution is almost optimal, except that it is restricted in certain ways. The main strategy will be modifying an optimal solution, so that it respects the same kind of constraints of a solution given by the algorithm. Also, the cost increase due to these modifications will be bounded by a small fraction of the optimal cost. In what follows, we assume that we have run Algorithm 1, giving as input an instance for the circle bin packing (\mathcal{C}, w, h) , and parameters $r \in \mathbb{Z}_+$, $\gamma > 0$.

First, we give some definitions that will be useful for the analysis. In each iteration, the algorithm considers small sub-bins, that are obtained by creating grids of elements with size $w_j \times h_j$, for some $j \geq 0$, over the larger bins of size $w \times h$. This notion is formalized in the following.

Definition 4. Consider a bin B of size $w_B \times h_B$. We say that B respects $w \times h$ if $w_B = w_j$, and $h_B = h_j$ for some $j \geq 0$. Also, if D is a set of bins, then we say that D respects $w \times h$ if every $B \in D$ respects $w \times h$.

Next, the function $\text{Gr}_j(B)$ will denote the set of elements in the grid of sub-bins of size $w_j \times h_j$ over one bin B .

Definition 5. Let $j \geq 0$. For a given bin B , we denote by $\text{Gr}_j(B)$ the set of elements of the grid of sub-bins of size $w_j \times h_j$ over B . Also, if D is a set of bins, then $\text{Gr}_j(D) = \cup_{B \in D} \text{Gr}_j(B)$.

We also need a definition to account the estimate number of bins of original size $w \times h$, that are needed to pack a set of sub-bins or circles.

Definition 6. If B is a rectangle or circle, then $N(B) = \text{Area}(B)/(wh)$. Also, if D is a set of rectangles or circles, then $N(D) = \sum_{B \in D} N(B)$.

We remark that, if a set of bins D respects $w \times h$, then bins of D can be easily combined into bins of size $w \times h$ using almost the same area. Thus, $N(D)$ is an estimate on the number of bins of size $w \times h$ needed to pack D . Indeed, if there is a packing of a set of circles C into a set of bins D that respects $w \times h$, then there is a packing of C in $\lceil N(D) \rceil$ bins of size $w \times h$.

Our algorithm also deals with bins of size $w_j \times (1 + \gamma)h_j$. Each of the definitions above has a counterpart for the augmented bins. We say that a bin B of size $w_B \times h_B$ respects $w \times (1 + \gamma)h$ if $w_B = w_j$ and $h_B = (1 + \gamma)h_j$ for some $j \geq 0$. Also, we decorate with a tilde the analogous versions of Gr and N , so $\tilde{\text{Gr}}_j(B)$ represents the grid of sub-bins of size $w_j \times (1 + \gamma)h_j$ over a bin B , and $\tilde{N}(B) = \text{Area}(B)/((1 + \gamma)wh)$.

4.2.1 Bounding the wasted area

The solutions obtained by Algorithm 1 may be suboptimal mostly because of the unused area. For instance, for some j , the area of bins in U_j is not fully used, since there might be elements of U_j that intersect circles of S_j only partially. This waste is bounded by the following lemmas.

Lemma 4. Let $C \in S_j$ be a circle packed into a bin B , and let $D \subseteq \tilde{\text{Gr}}_{j+1}(B)$ be the subset of bins in the grid that intersect circle C , but are not contained in C . Then $\tilde{N}(D) \leq 16\varepsilon N(C)$.

Proof. Let r_c be the radius of C . Each element of D has width $w_{j+1} = \varepsilon^{2(t+jr)+1}w$, and height $(1+\gamma)h_{j+1} = (1+\gamma)\varepsilon^{2(t+jr)+1}w$. Also, since $C \in S_j$, we have $2r_c \geq \varepsilon^{2(t+jr)}w$. Consider the circles C_+ and C_- , centered at the same point as C , and with radii $r_+ = r_c + w_{j+1} + (1+\gamma)h_{j+1}$ and $r_- = r_c - w_{j+1} - (1+\gamma)h_{j+1}$. Notice that every element of D is contained in $C_+ \setminus C_-$. We obtain

$$\begin{aligned} \text{Area}(D) &\leq \text{Area}(C_+) - \text{Area}(C_-) = \pi(r_+^2 - r_-^2) \\ &\leq ((1+2\varepsilon + (1+\gamma)2\varepsilon)^2 - (1-2\varepsilon - (1+\gamma)2\varepsilon)^2) \pi r_c^2 \\ &\leq (1+\gamma)16\varepsilon \text{Area}(C). \end{aligned}$$

Therefore, $\tilde{N}(D) = \text{Area}(D)/((1+\gamma)wh) \leq 16\varepsilon \text{Area}(C)/(wh) = 16\varepsilon N(C)$. \square

The next lemma is obtained analogously.

Lemma 5. *Let $C \in S_j$ be a circle packed into a bin B , and let $D \subseteq \text{Gr}_{j+1}(B)$ be the subset of bins in the grid that intersect circle C , but are not contained in C . Then $N(D) \leq 16\varepsilon N(C)$.*

4.2.2 Modifying an optimal solution

In the following, we will show that requiring that each set of circles S_j be packed into grid bins of size $w_j \times h_j$ does not increase much the solution cost. This fact is central to the algorithm, since it allows packing sets S_j 's iteratively. To show these properties, we will transform an optimal packing Opt of \mathcal{C} into a packing D with the desired properties. The idea is moving circles of S_j that intersect lines of the grid of size $w_j \times h_j$ to free bins that respect the grid.

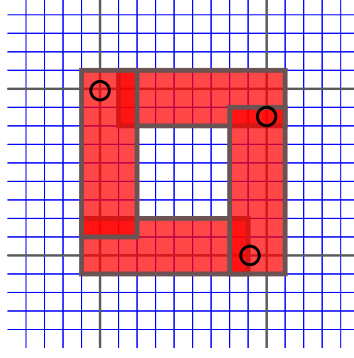
Notice that, in the optimal solution Opt , the circles of S_0 already respect the grid of size $w_0 \times h_0$, since $w = w_0$ and $h = h_0$. For each $j \geq 1$, we know that the diameter of a circle in S_j is at most εw_j , so the total area of circles in S_j that intersect the grid lines is $O(\varepsilon)\text{Area}(Opt)$. To fix the intersections, we may consider moving such circles to new bins of size $w_j \times h_j$ with area at most $O(\varepsilon)\text{Area}(Opt)$. As a first attempt, one could repeat this process for each set S_j , however, that would increase the solution cost by $O(\varepsilon)\text{Area}(Opt)$ for each $j \geq 1$. Therefore, we need a more involved approach: first, we move intersecting circles of S_1 to newly created sub-bins of size $w_1 \times h_1$; then, for each $j > 1$, we move the intersecting circles of S_j to sub-bins of size $w_j \times h_j$ placed over the space left by circles of S_{j-1} that intersected the grid lines, and that were moved in the previous iteration.

The next algorithm keeps the invariant that, at the start of iteration $j \geq 1$, the set R_j contains free space to pack all the grid intersecting circles of S_j . Steps (3a)-(3c) move intersecting circles to bins of R_j , and steps (3d)-(3f) prepare for the next iteration by making sure that there are enough free bins in R_{j+1} respecting the grid of size $w_{j+1} \times h_{j+1}$.

For $j \geq 1$, we will fix the intersections of S_j by considering each bin B of the grid $\text{Gr}_j(Opt)$ iteratively, in step (3b). We will consider 4 very thin rectangles over the boundary of B , as in Figure 3. The set of all such rectangles will be L_j . Here, we take advantage of the fact that the circles are non-overlapping, and move each rectangle containing the intersecting circles to the next free space of R_j , preserving the packing arrangement (alternatively, one could repack all intersecting circles using a different constant approximation algorithm).

To reserve free space in R_{j+1} for the next iteration, we will split the rectangles in L_j into sub-bins of size $w_{j+1} \times h_{j+1}$. We use as much as possible the free sub-bins of L_j . If a given sub-bin B of L_j is used by a circle, then we may use the sub-bin of Opt , denoted by $\phi(B)$, that was originally occupied by the corresponding region of this circle. There might be cases that the sub-bin B intersects the circle only partially, so that $\phi(B)$ is potentially used by other circles. In such cases, we create a new bin of size $w_{j+1} \times h_{j+1}$.

The complete algorithm to modify an optimal solution is formalized in the following.



Each circle of W has diameter at most εw_j . To rearrange the rectangles into bins of size $w_j \times w_j$ (of R_j), we use one side of length w_j . To ensure every circle is in a rectangle, the other side has length $3\varepsilon w_j$ (see lower circle).

Figure 3: Removing circles that intersect grid lines.

1. Let R_1 be a set of $12\varepsilon(wh)/(w_1h_1)|Opt|$ new bins of size $w_1 \times h_1$;
2. Let $D_0 = Opt \cup R_1$;
3. For each $j \geq 1$:

Move to R_j the circles in S_j that intersect grid lines:

- (a) Let $L_j = \emptyset$;
- (b) For each bin $B \in Gr_j(Opt)$:
 - i. Let W be the set of circles in S_j that intersect the boundary of B ;
 - ii. Let V be a set of 4 new bins (2 of size $3\varepsilon w_j \times h_j$, and 2 of size $w_j \times 3\varepsilon h_j$) placed over the boundary of B , so that each circle in W is contained in one bin of V (see Figure 3);
 - iii. For each cell $B' \in Gr_{j+1}(V)$, let $\phi(B')$ be the cell of $Gr_{j+1}(Opt)$ under B' ;
 - iv. Remove each circle of W from the packing D_j and pack it over one bin of V preserving the arrangement;
 - v. Add V to L_j ;
- (c) Make groups of $r/3$ bins (of equal sizes) of L_j forming new bins of size $w_j \times h_j$, and place each such bin over one distinct element of R_j ;

Reserve space for R_{j+1} :

- (d) Let $R_{j+1} = \emptyset$;
- (e) Let $N_j = \emptyset$;
- (f) For each bin $B \in Gr_{j+1}(L_j)$, consider the cases:
 - i. If B does not intersect any circle of S_j , then add B to R_{j+1} ;
 - ii. If B is contained in some circle of S_j , then add $\phi(B)$ to R_{j+1} ;
 - iii. If B intersects, but is not contained in a circle of S_j , then create a new bin of size $w_{j+1} \times h_{j+1}$ and add it to R_{j+1} , and to N_j ;
- (g) Let $D_j = D_{j-1} \cup N_j$;
- (h) If $\mathcal{C} \setminus H_t = S_0 \cup \dots \cup S_j$, make $D = D_j$, and stop.

We will show that D is a bin packing of $\mathcal{C} \setminus H_t$. First, we note that the procedure is well defined. It is enough to check that R_j has free space to pack bins of L_j . Indeed, it is not hard to show the following claim.

Claim 1. *For every $j \geq 1$, $\text{Area}(L_j) = \text{Area}(R_j) = \text{Area}(R_1)$.*

Proof. First, note that $\text{Area}(L_j) = \text{Area}(R_1)$ for every $j \geq 1$. Indeed, we have $|\text{Gr}_j(\text{Opt})| = |\text{Opt}|(wh)/(w_j h_j)$, and for each bin B in $\text{Gr}_j(\text{Opt})$, we create a set V such that $\text{Area}(V) = 4 \cdot 3\varepsilon w_j h_j$, so $\text{Area}(L_j) = 12\varepsilon(wh)|\text{Opt}| = \text{Area}(R_1)$.

Also, note that $\text{Area}(R_{j+1}) = \text{Area}(L_j)$, since for each $B \in \text{Gr}_{j+1}(L_j)$, $\text{Area}(R_{j+1})$ is increased by $\text{Area}(B)$. This is clear if steps (3(f)i) and (3(f)iii) are executed, so it is enough to show that, if step (3(f)ii) is executed for bins B, B' in $\text{Gr}_{j+1}(L_j)$, with $\phi(B) = \phi(B')$, then $B = B'$. By the definition in step (3(b)iii), B and B' must intersect the same region of a circle in S_j . Since each such circle is contained in exactly one rectangle of L_j , it follows that, indeed, $B = B'$. Then $\text{Area}(R_{j+1}) = \text{Area}(L_j)$.

Therefore $\text{Area}(R_j) = \text{Area}(R_1)$ for every $j \geq 1$. \square

Now, it will be shown that the algorithm produces a modified solution with the desired properties.

Claim 2. *At the end of iteration $j \geq 0$, the following statements hold:*

1. D_j is a packing of \mathcal{C} ;
2. for each $\ell = 0, \dots, j$, there is a packing of S_ℓ into a set $P'_\ell \subseteq \text{Gr}_\ell(D_j)$ of bins of size $w_\ell \times h_\ell$;
3. the bins in $R_{j+1} \subseteq \text{Gr}_{j+1}(D_j)$ do not intersect any circle of \mathcal{C} .

Proof. By induction on j . For $j = 0$, the statements are clear. So let $j \geq 1$, and assume that the statements are true for $j - 1$.

Statement 1: Clearly, L_j is a packing of the circles that were removed from the original packing D_{j-1} . Since r is a multiple of 3 and by Claim 1, step (3c) is well defined, and we can place rectangles of L_j over bins of R_j . After step (3c), we have a bin packing of \mathcal{C} , since, by the induction hypothesis, the set R_j did not intersect any circle at the beginning of iteration j . This shows statement 1.

Statement 2: Since, at the end of iteration, each circle of S_j that intersected a line of the grid $\text{Gr}_j(D_{j-1})$ is completely contained in a bin of $R_j \subseteq \text{Gr}_j(D_j)$, we obtain statement 2.

Statement 3: If step (3(f)i) or step (3(f)iii) is executed, then we add a free bin to R_{j+1} . Thus, we only need to argue that, whenever step (3(f)ii) is executed, the bin $\phi(B)$ does not intersect any circle. Let C be the circle that contains B , so that at the beginning of the iteration, $\phi(B)$ was contained in C . Since C was moved to L_j , $\phi(B)$ does not intersect any circle when step (3(f)ii) is executed. This completes the proof. \square

Finally, we may calculate the cost of the modified solution D .

Claim 3. $N(D) \leq (1 + 28\varepsilon) \text{OPT}_{w \times h}(\mathcal{C})$.

Proof. Let m be the number of iterations of the algorithm that modifies Opt . Notice that D is the disjoint union of Opt , R_1 , N_1, \dots, N_m . Thus we get

$$\begin{aligned} N(D) &= N(\text{Opt}) + N(R_1) + \sum_{j=1}^m N(N_j) \\ &\leq N(\text{Opt}) + 12\varepsilon(wh)|\text{Opt}|/(wh) + \sum_{j=1}^m 16\varepsilon N(S_j) \\ &\leq N(\text{Opt}) + 12\varepsilon N(\text{Opt}) + 16\varepsilon N(\mathcal{C}) \\ &\leq (1 + 28\varepsilon) \text{OPT}_{w \times h}(\mathcal{C}), \end{aligned}$$

where the first inequality comes from Lemma 5. \square

By combining the last claims, we obtain the following lemma.

Lemma 6. *There is a packing of $\mathcal{C} \setminus H_t$ into a set of bins D that respects $w \times h$ with $N(D) \leq (1 + 28\varepsilon) \text{OPT}_{w \times h}(\mathcal{C})$, such that for every $j \geq 0$, there is a packing of S_j into a set of bins $P'_j \subseteq \text{Gr}_j(D)$.*

In addition to requiring that each set of circles S_j be packed into bins of the grid with cells of size $w_j \times h_j$, we also require that the bins used to pack S_{j+1}, S_{j+2}, \dots do not intersect circles of S_1, \dots, S_{j-1} . Again, this restriction only increases the cost of a solution by a small fraction of the optimal value, as shown by the next lemma.

Lemma 7. *There is a packing of $\mathcal{C} \setminus H_t$ into a set of bins D that respects $w \times h$ with $N(D) \leq (1 + 44\varepsilon) \text{OPT}_{w \times h}(\mathcal{C})$, such that for every $j \geq 0$, there is a packing of S_j into a set of bins $P'_j \subseteq \text{Gr}_j(D)$. Moreover, if $B \in P'_j$, then B does not intersect any circle in S_ℓ for $\ell < j$.*

Proof. We start with a solution D' given by Lemma 6, and corresponding bin packings P'_j for each j . Without loss of generality, we assume that for each $B \in P'_j$, there is a circle $i \in S_j$ packed in B , since otherwise we could simply remove B from P'_j . We execute the following steps:

1. Let $P''_0 = P'_0$, and $D = D'$;
2. For each $j \geq 1$:
 - (a) Let $V \subseteq \text{Gr}_j(D)$ be the set of circles that intersect a circle $C \in S_{j-1}$, but are not contained in C .
 - (b) Create a set V' of $|V|$ new bins of size $w_j \times h_j$, and move the circles in V to V' preserving the arrangement.
 - (c) Let $P''_j = P'_j \cup V'$, and add V' to D .
 - (d) If $\mathcal{C} \setminus H_t = S_0 \cup \dots \cup S_j$, stop.

Clearly, the output of this procedure is a bin packing of \mathcal{C} , and a simple induction shows that at the end of iteration m , for every $j \leq m$, set $P''_j \subseteq \text{Gr}_j(D)$ is a bin packing of S_j , and for every $B \in P'_j$, B does not intersect any circle in S_ℓ for $\ell < j$. Using Lemma 5, we obtain

$$\begin{aligned} N(D) &\leq N(D') + \sum_{j \geq 0} 16\varepsilon N(S_j) \\ &\leq (1 + 28\varepsilon) \text{OPT}(\mathcal{C}) + 16\varepsilon \text{OPT}(\mathcal{C}) \\ &= (1 + 44\varepsilon) \text{OPT}(\mathcal{C}). \end{aligned} \quad \square$$

4.2.3 Obtaining an APTAS

Now we calculate the cost of the solution generated by Algorithm 1.

Lemma 8. *Algorithm 1 produces a packing of \mathcal{C} into bins of size $w \times (1 + \gamma)h$ using at most $(1 + O(\varepsilon)) \text{OPT}_{w \times h}(\mathcal{C}) + 2$ bins.*

Proof. Let D be the packing of \mathcal{C} obtained from Lemma 7, and let sets P'_j , $j \geq 0$, be the packings obtained for each S_j . Notice that for each $j \geq 0$, $\text{OPT}_{w_j \times h_j}(S_j) \leq |P'_j|$. Recall that Algorithm 1 uses Theorem 1 to obtain a packing P_j of S_j with $|P_j| \leq (1 + \varepsilon) \text{OPT}_{w_j \times h_j}(S_j)$ bins. Therefore, $|P_j| \leq (1 + \varepsilon)|P'_j|$, and thus $\tilde{N}(P_j) \leq (1 + \varepsilon)N(P'_j)$.

First, we show, by induction on m , that for every $m \geq 0$, we have $\tilde{N}(F_m) = \sum_{j=0}^{m-1} \tilde{N}(A_j) - \sum_{j=0}^{m-1} \tilde{N}(U_j)$. For $m = 0$, the claim is clear, so suppose that the claim is true for some $m \geq 0$. We have,

$$\begin{aligned} \tilde{N}(F_{m+1}) &= \tilde{N}(F_m \cup A_m) - \tilde{N}(U_m) \\ &= \tilde{N}(F_m) + \tilde{N}(A_m) - \tilde{N}(U_m) \\ &= \sum_{j=0}^{m-1} \tilde{N}(A_j) - \sum_{j=0}^{m-1} \tilde{N}(U_j) + \tilde{N}(A_m) - \tilde{N}(U_m) \\ &= \sum_{j=0}^m \tilde{N}(A_j) - \sum_{j=0}^m \tilde{N}(U_j). \end{aligned}$$

Now, we show that for every $m \geq 0$, we have $\sum_{j=0}^m \tilde{N}(A_j) \leq (1 + \varepsilon)N(D) + 15\varepsilon \sum_{j=0}^m N(S_j)$. Again, by induction on m . The case $m = 0$ is clear, so suppose that this is true for some $m \geq 0$. We consider two cases. If $|A_{m+1}| = 0$, then by the induction hypothesis we have

$$\begin{aligned} \sum_{j=0}^{m+1} \tilde{N}(A_j) &= \sum_{j=0}^m \tilde{N}(A_j) \\ &\leq (1 + \varepsilon)N(D) + 15\varepsilon \sum_{j=0}^m N(S_j) \\ &\leq (1 + \varepsilon)N(D) + 15\varepsilon \sum_{j=0}^{m+1} N(S_j). \end{aligned}$$

If $|A_{m+1}| > 0$, then we know that $\tilde{N}(A_{m+1}) + \tilde{N}(F_{m+1}) = \tilde{N}(P_{m+1})$, so we get

$$\begin{aligned} \sum_{j=0}^{m+1} \tilde{N}(A_j) &= \sum_{j=0}^m \tilde{N}(A_j) + \tilde{N}(A_{m+1}) \\ &= \left(\sum_{j=0}^m \tilde{N}(U_j) + \tilde{N}(F_{m+1}) \right) + \left(\tilde{N}(P_{m+1}) - \tilde{N}(F_{m+1}) \right) \\ &\leq (1 + 16\varepsilon) \sum_{j=0}^m N(S_j) + (1 + \varepsilon)N(P'_{m+1}) \\ &= (1 + \varepsilon) \left(\sum_{j=0}^m N(S_j) + N(P'_{m+1}) \right) + 15\varepsilon \sum_{j=0}^m N(S_j) \\ &\leq (1 + \varepsilon)N(D) + 15\varepsilon \sum_{j=0}^{m+1} N(S_j). \end{aligned}$$

The first inequality comes from Lemma 4, and the second inequality comes from the fact that bins in P'_{m+1} do not intersect circles in S_1, \dots, S_m .

It follows that the total area of bins used for circles in $\mathcal{C} \setminus H_t$ is

$$\begin{aligned} \sum_{j \geq 0} \tilde{N}(A_j) &\leq (1 + \varepsilon)N(D) + 15\varepsilon \sum_{j \geq 0} N(S_j) \\ &\leq (1 + \varepsilon)(1 + 44\varepsilon) \text{OPT}(\mathcal{C}) + 15\varepsilon \text{OPT}(\mathcal{C}) \\ &\leq (1 + 105\varepsilon) \text{OPT}(\mathcal{C}). \end{aligned}$$

For the circles in H_t , we used the NFDH algorithm to pack the bounding boxes of the circles, thus the density of the packing in each used bin of size $w \times (1 + \gamma)h$, with the exception of the last, is at least $1/4$. Since each circle occupies a fraction of $\pi/4$ of its bounding box, the total number of bins used for H_t is bounded by

$$\lceil 4 \cdot 4/\pi \tilde{N}(H_t) \rceil \leq \lceil 4 \cdot 4/\pi \varepsilon \tilde{N}(\mathcal{C}) \rceil \leq \lceil 16/\pi \varepsilon \text{OPT}(\mathcal{C}) \rceil.$$

Therefore, noticing that the sets of bins A_0, A_1, \dots respects $w \times (1 + \gamma)h$, we obtain that the total number of bins of size $w \times (1 + \gamma)h$ used to pack \mathcal{C} is

$$\lceil (1 + 105\varepsilon) \text{OPT}(\mathcal{C}) \rceil + \lceil 16/\pi\varepsilon \text{OPT}(\mathcal{C}) \rceil \leq (1 + 111\varepsilon) \text{OPT}(\mathcal{C}) + 2. \quad \square$$

Next lemma shows that running time of Algorithm 1 is polynomial if the ratio of a bin's dimensions is bounded by a constant.

Lemma 9. *Suppose $h/w = O(1)$. For any given constant r , and number $\gamma > 0$, the running time of Algorithm 1 is polynomial.*

Proof. Notice that in an implementation of Algorithm 1, we only run step (9) for nonempty sets S_j . In each such iteration, we only need to account for the running time of step (9a), and the running time to pack the bins of P_j in free space of current bins.

We consider the following equivalent and alternative procedure for step (9a):

1. scale the radius of each circle in S_j by $1/w_j$ and obtain a set S'_j ;
2. run algorithm of Theorem 1 with the scaled S'_j and bins of width 1 and height $h_j/w_j \leq h/w \in O(1)$, and obtain a packing P'_j ;
3. scale the obtained packing P'_j by w_j , and obtain a packing P_j .

Notice that for every j , the radius of the smallest circle in S'_j is at least a constant $\delta = \varepsilon^{2(t+jr)}w/(2w_j) = \varepsilon^{2(t+jr)}w/(2\varepsilon^{2(t+(j-1)r)+1}w) = \varepsilon^{2r-1}$, thus, Theorem 1 implies that the total running time of step (9a) is polynomial.

To pack a bin of P_j , for $j \geq 0$, we need to find one element $\tilde{\text{Gr}}_j(A_0 \cup \dots \cup A_{j-1})$ that is not in U_0, \dots, U_{j-1} , that is, we need to find a grid cell that does not intersect any circle of $S_0 \cup \dots \cup S_{j-1}$. To verify whether a grid cell intersects a circle of S_ℓ , with $0 \leq \ell \leq j-1$, it is enough to list the elements of U_ℓ that intersect the border of the circle. There is at most a constant number of such elements per circle, so at most $O(n)$ elements are listed until we find one free cell. Therefore, Algorithm 1 can be implemented in polynomial time. \square

Combining Lemmas 8 and 9 we obtain our main theorem.

Theorem 2. *Let (\mathcal{C}, w, h) be an instance of the circle bin packing. For any constant $\varepsilon > 0$, and number $\gamma > 0$, we can obtain in polynomial time a packing of \mathcal{C} into at most $(1 + \varepsilon) \text{OPT}_{w \times h}(\mathcal{C}) + 2$ rectangular bins of size $w \times (1 + \gamma)h$.*

Proof. If $h/w < 1/\varepsilon^2 \in O(1)$, then the theorem is immediate, so assume $h/w \geq 1/\varepsilon^2$. Consider an optimal solution Opt of bins of size $w \times h$. We will transform this solution into a packing of bins of size $w \times w/\varepsilon$. First, split each bin of Opt in sub-bins of size $w \times w/\varepsilon$. Then, remove all circles that intersect consecutive sub-bins. The total area of removed circles is at most $|\text{Opt}|(w \cdot 2w) \lfloor h/(w/\varepsilon) \rfloor \leq |\text{Opt}|2wh\varepsilon$. Finally, place the removed circles into their bounding boxes and pack them into additional bins of size $w \times w/\varepsilon$ using the NFDH strategy. Since each additional bin has density of at least $\pi/16$ (with exception of the last), the number of such bins is bounded by $\lceil (16/\pi |\text{Opt}|2wh\varepsilon)/(w(w/\varepsilon)) \rceil$. Therefore, we know that $\text{OPT}_{w \times w/\varepsilon}(\mathcal{C})$ is bounded by $|\text{Opt}| \lceil h/(w/\varepsilon) \rceil + \lceil (16/\pi |\text{Opt}|2wh\varepsilon)/(w(w/\varepsilon)) \rceil \leq (1 + O(\varepsilon))|\text{Opt}|(h/w)\varepsilon$, where we have used $h/w \geq 1/\varepsilon^2$ in the inequality.

Now, we use Lemma 8 and obtain a packing of \mathcal{C} into bins of size $w \times (1 + \gamma)w/\varepsilon$ of cost at most

$$(1 + O(\varepsilon)) \text{OPT}_{w \times w/\varepsilon}(\mathcal{C}) + 2 \leq (1 + O(\varepsilon))|\text{Opt}|(h/w)\varepsilon + 2.$$

By joining each group of $\lfloor h/(w/\varepsilon) \rfloor$ bins, we obtain a packing into bins of size $w \times (1 + \gamma)h$ of cost at most

$$\left\lceil \frac{(1 + O(\varepsilon))|Opt|(h/w)\varepsilon + 2}{\lfloor h/(w/\varepsilon) \rfloor} \right\rceil \leq (1 + O(\varepsilon))|Opt| + 2,$$

where the inequality follows from the fact that $h/w \geq 1/\varepsilon^2$, and assuming ε sufficiently small. To complete the proof, it is enough to notice that the running time is given by Lemma 9. \square

Now, it is straightforward to extend Theorem 2 to the circle strip-packing.

Theorem 3. *Let \mathcal{C} be a set of circles. For any given constant $\varepsilon > 0$, we can obtain in polynomial time a packing of \mathcal{C} in a strip of unit width and height $(1 + \varepsilon)\text{OPTS}(\mathcal{C}) + O(1/\varepsilon)$, where $\text{OPTS}(\mathcal{C})$ is the height of the minimum packing of \mathcal{C} in a strip of unit width.*

5 A Resource Augmentation Scheme for Circle Bin Packing

In this section, we discuss the use of augmented bins by our algorithm. The reason to use bins of height $1 + \gamma$ is that the algebraic algorithms used in Section 2 only give approximate packing with rational coordinates, and thus we enlarge the size of the bin to avoid intersections. Other approaches could be considered to obtain approximate packings, as well. One alternative would be discretizing the possible locations for circle centers. The idea is to create a grid of points, for which the distance between two adjacent locations is γ , for some small $\gamma > 0$. Then, we could modify an optimal solution so that the center of each center is moved to the closest point in the grid, and obtain an $O(\gamma)$ -packing. To solve the problem of packing a constant-sized set of circles in an augmented bin, we could simply try all possible combinations, and check whether we obtain an $O(\gamma)$ -packing.

Notice that this discretization algorithm does not provide a certificate that there exists a packing into the original (non-augmented) bin, that is, we may obtain an approximate packing for a set of circles that cannot be packed in a bin. When using a quantifier elimination algorithm, we obtain such a certificate, and the need for approximate packings is due to numeric reasons only. Moreover, algebraic algorithms have a much weaker dependency on the size of the resource augmentation parameter, $\log 1/\gamma$. Namely, algebraic algorithms have time complexity $O((\log 1/\gamma)^{O(1)})$, while discretization algorithms would have time complexity $O((1/\gamma)^{O(1)})$. This means that, if γ is part of the input, then the algebraic algorithms would be polynomial, while the discretization algorithms would not.

Again, the sole reason for having augmented bins in Theorem 2 is that we insisted in obtaining a rational solution. Thus, on the one hand, if we allowed a more general model of computation, in which one could compute and operate over polynomial solutions, we would obtain an APTAS for the circle bin packing problem without resource augmentation. On the other hand, if we admit resource augmentation, and consider the particular case that a bin is enlarged by a constant size, then we may strengthen Theorem 2, and obtain a packing with no more bins than in an optimal solution, as in the following result.

Theorem 4. *Let (\mathcal{C}, w, h) be an instance of the circle bin packing. For any given constant $\varepsilon > 0$, we can obtain in polynomial time a packing of \mathcal{C} into at most $\text{OPT}_{w \times h}(\mathcal{C})$ rectangular bins of size $w \times (1 + \varepsilon)h$.*

A few sources are responsible for the additional bins used in the algorithm of Theorem 2. In Theorem 1, which was used for the packing of large circles, we spent additional bins for the first group of Q circles. In Algorithm 1, the elements in the set of intermediate circles H_t are packed in separate new bins. Also, we wasted the space of sub-bins that partially intersected

large circles. In the following, we will discuss how to slightly modify our algorithm, and avoid the use of additional bins for each case. This will be done by augmenting the bin with small strips of height $O(\varepsilon)h$, thus obtaining Theorem 4.

First we consider an alternative approach for the packing of large circles of Theorem 1. Instead of mapping large circles to small circles, we simply round down the radius of each circle, and obtain an approximate packing.

Lemma 10. *Let (\mathcal{C}, w, h) be an instance of the circle bin packing, such that $w, h \in O(1)$, and $\min_{1 \leq i \leq n} r_i \geq \delta$, for some constant δ . For any given constant $\varepsilon > 0$, there is a polynomial-time algorithm that packs \mathcal{C} into at most $\text{OPT}_{w \times h}(\mathcal{C})$ rectangular bins of size $w \times (1 + \varepsilon)h$.*

Proof. Let $M = \lceil wh/\text{Area}(\delta) \rceil$ and $\alpha = \varepsilon^2/(6M^2)$. We obtain a modified instance \mathcal{C}' such that the radius of each circle is rounded down to a number in the sequence $\delta, \delta + \alpha, \delta + 2\alpha, \dots$. Thus, in \mathcal{C}' , the number of different radii is at most a constant. We use Lemma 3, and obtain a packing into at most $\text{OPT}_{w \times h}(\mathcal{C}') \leq \text{OPT}_{w \times h}(\mathcal{C})$ bins of size $w \times (1 + \varepsilon)h$. By restoring the original radii, we get a 2α -packing of \mathcal{C} . Finally, we use Lemma 2, and obtain a packing of \mathcal{C} into no more than $\text{OPT}_{w \times h}(\mathcal{C})$ bins of size $w \times (1 + O(\varepsilon))h$. \square

Now we proceed to prove Theorem 4.

Proof of Theorem 4. First, we pack the circles of H_t into at most $\text{OPT}(\mathcal{C})$ strips of height $O(\varepsilon)h$. We assume, without loss of generality, that $t > 0$, since otherwise we could modify the algorithm and find some $t' > 0$, such that $\text{Area}(H_{t'}) \leq 2\varepsilon \text{Area}(\mathcal{C})$. Since $t > 0$, each circle of H_t has radius at most εh . Therefore, we can pack H_t into strips of size $w \times 11\varepsilon h$ using the NFDH strategy. Using density arguments, we obtain that the total number of strips is

$$\begin{aligned} \lceil 4 \cdot 4/\pi \cdot \text{Area}(H_t)/(11\varepsilon wh) \rceil &\leq \lceil 4 \cdot 4/\pi \cdot 2\varepsilon \text{Area}(\mathcal{C})/(11\varepsilon wh) \rceil \\ &\leq \lceil 32/(11\pi) \text{OPT}(\mathcal{C}) \rceil \leq \text{OPT}(\mathcal{C}). \end{aligned}$$

Now, we only need few more changes to Algorithm 1. First, we set parameter $\gamma = \varepsilon$. Then, in step 9a, we replace Theorem 1 by Lemma 10. Finally, in step 10, rather than packing sub-bins of A_0, A_1, \dots into bins of size $w \times (1 + \varepsilon)h$, we pack A_0, A_1, \dots into bins of size $w \times (1 + 105\varepsilon)(1 + \varepsilon)h$. Notice that the simple first-fit greedy algorithm that packs such sub-bins in decreasing order of height has the property that, if a new bin is created to pack a sub-bin B , then either $B \in A_0$; or $B \notin A_0$ and every other created bin is fully used. Therefore, it is not hard to see that the total number of bins is bounded by $\max\{|A_0|, \lceil \sum_{j \geq 0} \text{Area}(A_j)/((1 + 105\varepsilon)(1 + \varepsilon)wh) \rceil\}$.

By Lemma 10, we may bound the first term as $|A_0| \leq \text{OPT}(S_0) \leq \text{OPT}(\mathcal{C})$. To bound the second term, we may repeat the proof of Lemma 8, and obtain

$$\sum_{j \geq 0} \tilde{N}(A_j) \leq (1 + 105\varepsilon) \text{OPT}(\mathcal{C}),$$

which implies

$$\sum_{j \geq 0} \text{Area}(A_j) \leq (1 + \varepsilon)(1 + 105\varepsilon)wh \text{OPT}(\mathcal{C}).$$

Therefore

$$\left\lceil \frac{\sum_{j \geq 0} \text{Area}(A_j)}{(1 + 105\varepsilon)(1 + \varepsilon)wh} \right\rceil \leq \left\lceil \frac{(1 + \varepsilon)(1 + 105\varepsilon)wh \text{OPT}(\mathcal{C})}{(1 + 105\varepsilon)(1 + \varepsilon)wh} \right\rceil = \text{OPT}(\mathcal{C}).$$

To complete the proof, we combine the bins used for A_0, A_1, \dots with the strips used for H_t . \square

6 Generalizations

In this section, we discuss possible generalizations to our algorithm that preserve the approximation ratio. We notice that, although Algorithm 1 only considers the case of circles, it can be applied to many other packing problems. Indeed, Algorithm 1 induces a *unified framework* for packing problems that satisfy certain assumptions, such as the existence of a polynomial-time algorithm to approximately pack “large” items (in an augmented bin), and bounded wasted volume of sub-bins that partially intersect an item.

As an illustrative example, in this section, we will consider the problem of packing d -dimensional L_p -norm spheres¹ in d -dimensional boxes, and give a high-level summary of necessary changes in the algorithm. Moreover, we will discuss how our algorithm can be used for bins of different shapes, provided that we can satisfy some conditions.

6.1 Packing d -dimensional L_p -norm spheres

For a vector $v \in \mathbb{R}^d$, we denote by $v(i)$ the i -th coordinate of v . A d -dimensional box of size v is a hyperrectangle with sides of length $v(i) > 0$, for each $1 \leq i \leq d$. In this section, we consider the problem of packing L_p -norm spheres in d -dimensional boxes of a given size, for a rational $p \geq 1$. In fact, we allow a slightly more general concept of norm, that we call the *weighted L_p -norm*, when each dimension can be stretched by a given factor. This is formally defined below.

Definition 7. Let d be a positive integer, p be a positive rational, and $\omega \in \mathbb{R}^d$ be a vector such that $\omega(i) \geq 1$, for $1 \leq i \leq d$. The weighted $L_{p,\omega}$ - and $L_{\infty,\omega}$ -norms are defined respectively as

$$\|x\|_{p,\omega} = \left(\sum_{i=1}^d \omega(i) |x(i)|^p \right)^{1/p}, \quad \text{and} \quad \|x\|_{\infty,\omega} = \max_{1 \leq i \leq d} \omega(i) |x(i)|.$$

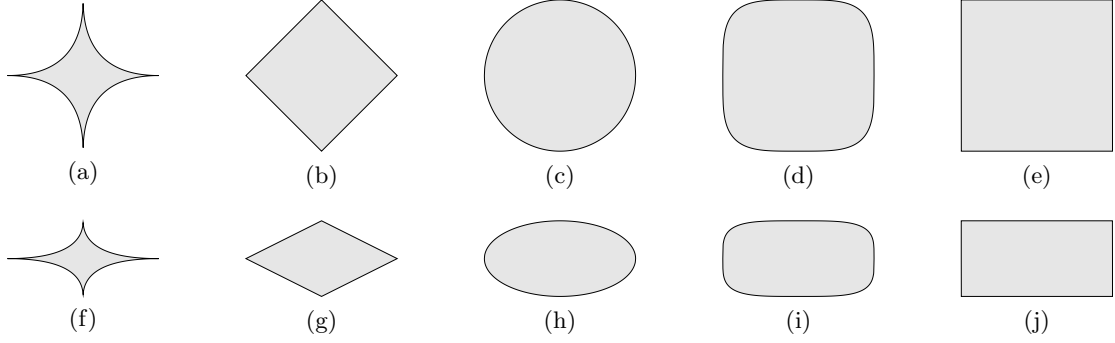
Also, let $r \in \mathbb{R}_+$, and $c \in \mathbb{R}^d$. The d -dimensional $L_{p,\omega}$ -norm sphere of radius r and centered at c is the set of points $x \in \mathbb{R}^d$ such that $\|x - c\|_{p,\omega} < r$. The d -dimensional $L_{\infty,\omega}$ -norm sphere is defined analogously.

Figure 4 contains examples of the considered spheres. Notice that the assumption that $\omega(i) \geq 1$, for $1 \leq i \leq d$, ensures that each sphere of radius r fits in a box with sides of length $2r$. This is without loss of generality, since we could normalize the radii of the spheres otherwise. Also, we will consider only the case that $p \geq 1$. This restriction implies that the spheres correspond to convex regions in the d -dimensional space (see Figures 4(a) and 4(f) for examples when $p < 1$).

Now we show how to pack d -dimensional $L_{p,\omega}$ -norm spheres, for given d , p and ω . For that, we need three main steps:

1. obtain an algorithm that decides whether a set of spheres can be packed in a given box, and provides a packing within an arbitrarily small error precision;
2. show how to transform this approximate packing into a non-intersecting packing in an augmented bin;
3. and show that the wasted volume after discarding bins that partially intersect spheres is a small factor of the spheres' volume.

¹To comply with the majority of works in the literature, in this paper we use the term *circle*, rather than *disk*, to refer to the interior of a region. Similarly, and for the sake of consistency with the multidimensional packing literature, we use the term *sphere*, rather than *ball*, to refer to the interior of a solid.



In order, the columns correspond to $p = 0.5, 1, 2, 4, \infty$, and the rows correspond to $\omega = (1, 1), (1, 2)$. All spheres have radius 1.

Figure 4: Examples of 2-dimensional $L_{p,\omega}$ -spheres.

First, consider the case that p is rational, and let a and b be positive integers such that $p = a/b$. Also, let $v \in \mathbb{R}_+^d$ be the box size, and $\omega \in \mathbb{R}_+^d$ be the norm weight. We want to obtain an algorithm similar to that of Corollary 1. The convexity of the spheres implies that deciding if a set $\mathcal{C} = \{1, \dots, n\}$ of $L_{\omega,p}$ -spheres of radii r_1, r_2, \dots, r_n can be packed in a box of size v can be encoded by the following system of inequalities, where variables $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ correspond to centers of the spheres.

$$\sum_{k=1}^d (\omega(k) |x_i(k) - x_j(k)|)^p \geq (r_i + r_j)^p \quad \text{for } 1 \leq i < j \leq n, \quad \text{and} \quad (8)$$

$$r_i \leq x_i(k) \leq v(k) - r_i \quad \text{for } 1 \leq i \leq n, \quad 1 \leq k \leq d. \quad (9)$$

In order to transform (8)-(9) into a polynomial system, we have to deal with two problems: avoiding the modulus operator, and dealing with non-integer exponents. Consider an expression e . We can replace $|e|$ by a new variable z , if we add the constraints $z^2 = e^2$, and $z \geq 0$. Also, if $e \geq 0$, we can replace $e^{a/b}$ by a new variable y , by adding the constraints $e^a = y^b$, and $y \geq 0$. Therefore, we can transform (8)-(9) into a system of polynomial inequalities and equalities by adding new variables y_{ij} and z_{ij} in \mathbb{R}^d for every $1 \leq i < j \leq n$, and considering the following system.

$$\begin{aligned} \sum_{k=1}^d y_{ij}(k) &\geq (r_i + r_j)^p && \text{for } 1 \leq i < j \leq n, \\ r_i &\leq x_i(k) \leq v(k) - r_i && \text{for } 1 \leq i \leq n, \quad 1 \leq k \leq d, \\ y_{ij}(k)^b &= (\omega(k) z_{ij}(k))^a && \text{for } 1 \leq i < j \leq n, \quad 1 \leq k \leq d, \\ z_{ij}(k)^2 &= (x_i(k) - x_j(k))^2 && \text{for } 1 \leq i < j \leq n, \quad 1 \leq k \leq d, \quad \text{and} \\ y_{ij}(k), z_{ij}(k) &\geq 0 && \text{for } 1 \leq i < j \leq n, \quad 1 \leq k \leq d. \end{aligned}$$

We can solve this system in a way similar to Corollary 1, and obtain an arrangement of the spheres that intersect by at most a small value. Now, we modify this arrangement to obtain a feasible packing in an augmented box, like in Lemma 1. The d -dimensional box will be augmented only in its first dimension. Therefore, to avoid intersection of two spheres, one of these will have the first coordinate shifted. The following lemma gives a bound on the shift length that is necessary.

Lemma 11. Let r_1, r_2, h, ε be positive numbers such that $\varepsilon h \leq r_1 + r_2 \leq h$, and $\varepsilon < 1$, and let x_1, x_2 be points in the d -dimensional space. If $x_1(1) \geq x_2(1)$, $\|x_1 - x_2\|_{p,\omega} \geq r_1 + r_2 - \varepsilon^2 h$, and $x'_1 = (x_1(1) + t, x_1(2), \dots, x_1(d))$, where $t = (2^a \varepsilon)^{1/p} h / \omega(1)$, then $\|x'_1 - x_2\|_{p,\omega} \geq r_1 + r_2$.

Proof. By direct calculation,

$$\begin{aligned}
\|x'_1 - x_2\|_{p,\omega} &= \left(\sum_{k=1}^d (\omega(k) |x'_1(k) - x_2(k)|)^p \right)^{1/p} \\
&= \left((\omega(1) |x'_1(1) - x_2(1)|)^p + \sum_{k=2}^d (\omega(k) |x'_1(k) - x_2(k)|)^p \right)^{1/p} \\
&= \left((\omega(1) |x_1(1) + t - x_2(1)|)^p + \sum_{k=2}^d (\omega(k) |x_1(k) - x_2(k)|)^p \right)^{1/p} \\
&= \left((\omega(1) |x_1(1) - x_2(1)| + \omega(1)t)^p + \sum_{k=2}^d (\omega(k) |x_1(k) - x_2(k)|)^p \right)^{1/p} \\
&\geq \left((\omega(1) |x_1(1) - x_2(1)|)^p + t^p \omega(1)^p + \sum_{k=2}^d (\omega(k) |x_1(k) - x_2(k)|)^p \right)^{1/p} \\
&= ((\|x_1 - x_2\|_{p,\omega})^p + t^p \omega(1)^p)^{1/p} \\
&\geq ((r_1 + r_2 - \varepsilon h)^p + t^p \omega(1)^p)^{1/p} \\
&\geq ((r_1 + r_2 - \varepsilon(r_1 + r_2))^p + t^p \omega(1)^p)^{1/p} \\
&= ((1 - \varepsilon)^p (r_1 + r_2)^p + t^p \omega(1)^p)^{1/p}
\end{aligned}$$

We now bound t to obtain the desired result. First, notice that because $1 - \varepsilon < 1$, we have that $1 - (1 - \varepsilon)^p \leq 1 - (1 - \varepsilon)^a$, and so

$$\begin{aligned}
1 - (1 - \varepsilon)^a &= 1 - \left(1 + \sum_{i=1}^a \binom{a}{i} (-\varepsilon)^i \right) = - \sum_{i=1}^a \binom{a}{i} (-\varepsilon)^i \\
&\leq \sum_{i=1}^a \binom{a}{i} \varepsilon^i \leq \varepsilon \sum_{i=1}^a \binom{a}{i} < \varepsilon 2^a,
\end{aligned}$$

from where we conclude that $1 - (1 - \varepsilon)^p < 2^a \varepsilon$. Now, using the definition of t , we have that

$$t = \frac{(2^a \varepsilon)^{1/p} h}{\omega(1)} \geq \frac{(1 - (1 - \varepsilon)^p)^{1/p} h}{\omega(1)} = \left(\frac{(1 - (1 - \varepsilon)^p) h^p}{\omega(1)^p} \right)^{1/p}.$$

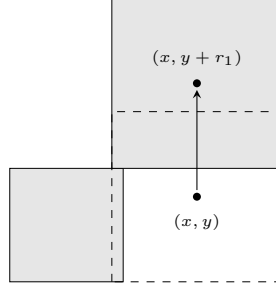
That is, $t^p \omega(1)^p \geq (1 - (1 - \varepsilon)^p) h^p \geq (1 - (1 - \varepsilon)^p) (r_1 + r_2)^p$. Thus, we conclude that

$$\begin{aligned}
\|x'_1 - x_2\|_{p,\omega} &\geq ((1 - \varepsilon)^p (r_1 + r_2)^p + t^p \omega(1)^p)^{1/p} \\
&\geq ((1 - \varepsilon)^p (r_1 + r_2)^p + (1 - (1 - \varepsilon)^p) (r_1 + r_2)^p)^{1/p} = r_1 + r_2,
\end{aligned}$$

and the results follows. \square

For the case $p = \infty$, we cannot repeat the same lifting strategy to obtain a packing in an augmented bin. As an example, even if two spheres intersect in a small portion, it could be necessary to raise one of them by twice the radius of the other sphere (see Figure 5). Therefore, instead of using algebraic quantifier elimination algorithm to pack a set of spheres in a given box, we can use an algorithm based on discretization. This has been already done, for example, by Bansal *et al.* [3] for hypercubes, and the generalization is straightforward.

Now, what is missing to extend Algorithm 1 is bounding the wasted volume of grid elements that partially intersect spheres. The following lemma bounds the distance between any two points in a grid element of side length ℓ . Then, we observe that the grid elements that are wasted must intersect the border of a packed sphere.



Fixing intersection of squares by lifting fails even if the intersection is small. In this example, one needs to raise one square by the length of its radius to avoid intersection. The square on the left has radius 0.5, and the square on the right has radius 0.75. To avoid intersection, we have to raise the square on the right by 1.

Figure 5: An example where lifting fails.

Lemma 12. *If x and y are two points in a hyperrectangle of side length ℓ , then*

$$\|x - y\|_{p,\omega} \leq \ell(\sum_{k=1}^d \omega(k)^p)^{1/p}, \quad \text{and} \quad \|x - y\|_{\infty,\omega} \leq \ell(\max_{1 \leq k \leq d} \omega(k)).$$

Proof. Note that, for every $1 \leq k \leq d$, $|x(i) - y(i)| \leq \ell$. If $p \in \mathbb{Q}$, then we get

$$\begin{aligned} \|x - y\|_{p,\omega} &= \left(\sum_{k=1}^d (\omega(k)|x(i) - y(i)|)^p \right)^{1/p} \\ &\leq \left(\sum_{k=1}^d (\omega(k)\ell)^p \right)^{1/p} = \ell \left(\sum_{k=1}^d \omega(k)^p \right)^{1/p}. \end{aligned}$$

Now, if $p = \infty$, we get

$$\begin{aligned} \|x - y\|_{p,\omega} &= \max_{1 \leq k \leq d} (\omega(k)|x(i) - y(i)|) \\ &\leq \max_{1 \leq k \leq d} \omega(k)\ell = \ell \max_{1 \leq k \leq d} \omega(k). \end{aligned}$$

□

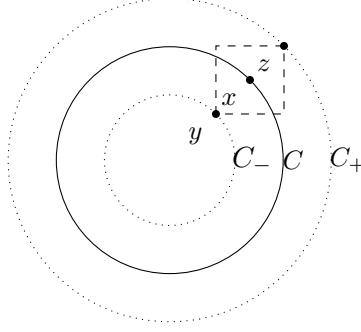
Lemma 13. *Let Q be a hyperrectangle of side length ℓ , and define*

$$t = \begin{cases} \ell(\sum_{k=1}^d \omega(k)^p)^{1/p} & \text{if } p \in \mathbb{Q}, \\ \ell(\max_{1 \leq k \leq d} \omega(k)) & \text{if } p = \infty. \end{cases}$$

Also let C be an $L_{p,\omega}$ -norm sphere of radius r centered at a point c , such that $r \geq t$, C_+ be the $L_{p,\omega}$ -norm sphere of radius $r + t$ centered at c , and C_- be the interior of the $L_{p,\omega}$ -norm sphere of radius $r - t$ centered at c . If C intersects Q , but Q is not contained in C , then $Q \subseteq C_+$ and $Q \cap C_- = \emptyset$.

Proof. Since $C \cap Q \neq \emptyset$, and $Q \not\subseteq C$, there exists $x \in Q$ such that $\|c - x\|_{p,\omega} = r$. Also, consider an arbitrary point $y \in Q$ (see Figure 6). Using the triangle inequality, and Lemma 12, we have that

$$\|c - y\|_{p,\omega} \leq \|c - x\|_{p,\omega} + \|x - y\|_{p,\omega} \leq r + t,$$



Point x is at the boundary of C . The dashed box has side 2ℓ , and is centered at x . Point y is at a distance $\ell\sqrt{2}$ of x , and is not contained in C_- . Similarly, z is not contained in C_+ . For this configuration, every box of side ℓ that intersects x must be contained in the region delimited by the dashed square.

Figure 6: Bounding the box intersection to a sphere boundary.

from where we conclude that $y \in C_+$, and hence $Q \subseteq C_+$. Again from the triangle inequality, and Lemma 12, we get

$$r = \|c - x\|_{p,\omega} \leq \|c - y\|_{p,\omega} + \|x - y\|_{p,\omega} \leq \|c - y\|_{p,\omega} + t,$$

so $\|c - y\|_{p,\omega} \geq r - t$, and we conclude that $y \notin C_-$. Thus, we get $Q \cap C_- = \emptyset$. \square

From Lemma 13, one may obtain a statement similar to Lemma 4. Using arguments of Section 4, combined with the discussion in this subsection, it is now possible to obtain an asymptotic approximation scheme for the problem of packing $L_{\omega,p}$ -spheres in the minimum number of d -dimensional bins of a given size. It is straightforward to extend these results for the strip packing problem of $L_{\omega,p}$ -spheres, when the recipient contains an unbounded dimension, and the objective is to minimize span of packed spheres in this dimension.

We finish this subsection by noticing that the bin packing of $L_{\infty,\omega}$ -norm spheres is, in fact, a particular case of the general hyperrectangle bin packing problem when all hyperrectangles are congruent.

6.2 Generalizing the bin

In this subsection, we sketch how our results can be generalized to deal with bins of different shapes, that are described by semi-algebraic sets. This generalization includes, for example, polytopes and $L_{p,\omega}$ -norm spheres. The idea is solving the corresponding system of polynomial inequalities, together with the packing constraints. We will consider a bin B that corresponds to the d -dimensional semi-algebraic set described by polynomials $f_1, \dots, f_s \in \mathbb{Q}[z_1, \dots, z_d]$. More precisely, a point $z \in \mathbb{R}^d$ is contained in B if, and only if, $f_i(z) \geq 0$ for all $1 \leq i \leq s$. In the following, we will derive a quantified formula with polynomial inequalities and equalities to decide if a set \mathcal{C} of spheres can be packed in B .

We begin by defining variables $x_1, \dots, x_n \in \mathbb{R}^d$ to represent the centers of the spheres. Consider the inequalities

$$\|x_i - x_j\|_{p,\omega} \geq r_i + r_j, \quad \text{for } 1 \leq i < j \leq n, \quad (10)$$

$$f_j(x_i) \geq 0, \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq s. \quad (11)$$

These constraints requires that there exists a feasible arrangement of the spheres in the d -dimensional space, and that each center is contained in the bin. Now, we will ensure that

every point of the sphere is indeed in the bin. To do this, for each sphere i , we will consider a mapping from the $(d+1)$ -dimensional space to this sphere, centered at x_i , that is, the function $g_i : \mathbb{R}^d \setminus \{0\} \times \mathbb{R} \rightarrow \mathbb{R}^d$ defined as

$$g_i(a, \lambda) = r_i \frac{a}{\|a\|_{p,\omega}(1 + \lambda^2)} + x_i.$$

This maps any pair (a, λ) of the domain to a point $y_i = g_i(a, \lambda)$ in the $L_{p,\omega}$ -sphere centered at x_i with radius r_i , or at its boundary. Analogously, each point in the sphere, with exception of the center, is mapped to a point of the domain. Therefore, it is enough to make sure that every such mapped point y_i is in the bin. This is done by adding the following constraints.

$$\|a\|_{p,\omega}(1 + \lambda^2)(y_i(k) - x_i(k)) = r_i a(k), \quad \text{for } 1 \leq i \leq n, 1 \leq k \leq d, \quad (12)$$

$$f_j(y_i) \geq 0, \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq s. \quad (13)$$

Notice that if $\|a\|_{p,\omega} = 0$, then inequality (12) is trivially satisfied, since in this case we have $a(k) = 0$ for every $1 \leq k \leq d$, so y_i can be set as any point of the bin to satisfy inequality (13). Now, conjoining all the constraints, we obtain a quantified first-order formula. Therefore, deciding if there is a packing of \mathcal{C} in B is equivalent to deciding the formula

$$(\exists x_1 \in \mathbb{R}^d) \dots (\exists x_n \in \mathbb{R}^d)(\forall a \in \mathbb{R}^d)(\forall \lambda \in \mathbb{R})(\exists y_1 \in \mathbb{R}^d) \dots (\exists y_n \in \mathbb{R}^d) \text{ (10)-(13)}.$$

Although constraints (10) and (12) contain non-polynomial terms, it is possible to obtain an equivalent formula with only polynomial inequalities and equalities using the process discussed before. Therefore, one can use any quantifier elimination algorithm to solve it, provided that it supplies a realization of the points at a given precision.

Also in this case, the obtained center coordinates can be irrational, so only approximate values may be available. To use resource augmentation, here we will scale both the recipient and the radii of the spheres by a factor $1 + \gamma$ before solving the algebraic system. This is done by replacing polynomial f_i by polynomial $f'_i = f_i(x/(1 + \gamma))$, for $1 \leq i \leq s$, and radius r_i by $r'_i = (1 + \gamma)r_i$, for $1 \leq i \leq n$. After obtaining an approximate packing, we scale back the radii of the spheres to their original values. The precision of the obtained packing is adjusted according to parameter γ , so that intersection is avoided.

Notice that after the first iteration of the packing algorithm, the remaining space of the bin will be partitioned into a grid of boxes. Therefore, the volume of the boxes that intersect the boundary of the bin will be lost. This is not a problem because we use a fine-grained grid, for which the wasted volume will be bounded by a small fraction of the bin volume.

7 Final Remarks

We presented the first approximation algorithms for the circle bin packing problem using augmented bins, and the circle strip packing problem. We obtained asymptotic approximation schemes for circle packings exploring novel ideas, such as iteratively distinguishing large and small items, and carefully using the free space left after packing large items. We believe that our algorithm can lead to further results for related problems, and we have already presented some possible generalizations. Also, our use of algebraic quantifier elimination algorithms exemplifies how results from algebra can be successfully used in the context of optimization. Using these algorithms helped us to avoid discretization algorithms, whose running time would depend exponentially on the size of resource augmentation parameter, $\log 1/\gamma$, and allowed the packing of more general items, in a simple and concise manner.

We highlight that our algorithm is not restricted to circles, and indeed it can be seen as a unified framework to different packing problems. These problems need to satisfy certain conditions, such as requiring that items may be partitioned into groups of large and small items (as in Figure 1), and the packing of large items in an augmented bin can be approximately solved. One may consider, for example, the problem of packing regular polygons, by using a discretization algorithm to deal with large items. Notice that one could even consider instances with items of mixed shapes. A very natural generalization of our algorithm is considering the packing of L_p -norm spheres, as done in Section 6. Moreover, minor modifications of the algorithm allow using bins of different shapes. In Section 6, the illustrative example considers a whole set of semi-algebraic sets as possible bins, what comprises, for instance, the problem of packing spheres in spheres. For the particular case that a bin is enlarged by a constant, we avoid the approximation, and obtain a resource augmentation scheme.

Finally, we note that, although the quantifier elimination algorithms we used give a precise representation of a packing in a non-augmented bin, the returned solution may possibly contain irrational coordinates. To provide solutions with rational numbers, we used approximate coordinates with arbitrary precision. This is the *only* reason why we used augmented bins in our APTAS, and thus resource augmentation can be avoided in a more general computational model. We left open the question to determine if it is always possible to obtain a rational solution to the problem of packing a set of circles of rational radii in a non-augmented bin of rational dimensions.

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